Estimating parameters of a nonlinear dynamical system

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A method based on a modified Newton-Raphson scheme is presented to estimate parameters of a nonlinear dynamical system from the time series data of the variables. The method removes some of the problems associated with the standard synchronization based methods. An important achievement of this method is that it is possible to determine the exact form of dynamical equations for systems with quadratic nonlinearity.

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Given the time series data, an important unsolved problem is to determine the form of the underlying dynamical equations. Since this problem is difficult to handle, one uses some physical model to determine the form of these equations. Recently, there has been considerable interest in determining parameters of a nonlinear chaotic dynamical system from the time series data given the form of the equations [1–4]. The interest is due to several possible applications as discussed in the literature. Most of these studies use the synchronization property of coupled dynamical systems [5,6]. One constructs a slave system coupled to the original system (master) using the known time series data obtained from the master. The methods to determine the parameters may be broadly categorized into two types. In the first type, one uses some way of minimizing the synchronization error, [1] while in the second type, one introduces additional dynamical evolution equations for the parameters [2].

Though the methods based on synchronization have proved to be very useful in estimating the parameters of nonlinear dynamical systems, two limitations of these methods may be noted. First, variables used in the estimation must be synchronizing variables in the sense that with a suitable coupling between two identical dynamical systems they should lead to synchronization. Second, the total time must be larger then the synchronizing time scale. The purpose of the present Brief Report is to present a method of parameter estimation which addresses these limitations. The method is based on a modification of the Newton-Raphson method to include dynamics [7–9]. The method is able to remove the above two problems associated with synchronization based methods and also the accuracy of parameter estimation is better. It works in the presence of noise. It is demonstrated using two examples. In particular, for a three dimensional system with quadratic nonlinearity one is able to determine the exact form of dynamical equations by determining all possible parameters of such a system.

Consider an autonomous dynamical system,

$$\dot{x} = f(x,\mu), \tag{1}$$

where $x=(x_1,\ldots,x_d)$ is a d-dimensional state vector, $f=(f_1,\ldots,f_d)$ and $\{\mu\}_j, j=1,\ldots,m$, are a set of m parameters.

As in the methods based on synchronization, assume that the equations of the dynamical system, i.e., the functional form of f and the time evolution of the variables x, are known [10]. Now suppose that the set of parameters μ is not known. (Known parameters are not included in μ.) Formally, the problem at hand consists of estimating the unknown parameters μ using the time series data.

Let y denote a system of variables, $y=(y_1,\ldots,y_d)$ such that y has an identical form of evolution to that of x (Eq. (1)) but with different values ν of m parameters,

$$\dot{y} = f(y,\nu). \tag{2}$$

Let $w(t)$ denote the difference $w(t)=y(t)-x(t)$. We look for the solution of the equation

$$w(t) = 0. \tag{3}$$

Noting that the initial-state vectors, $y(0)$ and $x(0)$ and the parameter sets μ and ν uniquely determine the difference $w(t)$, one solution of Eq. (3) is $y(0)=x(0)$ and $\nu=\mu$. Since $x(t)$ is assumed to be known, one can set $y(0)=x(0)$, and hence the solution of interest is $\mu=\nu$.

We now introduce the notation $w^n=w(n\Delta t)=w^n((y^0,\nu),(x^0,\nu))$, where Δt is a small time interval. Similarly, $y^n=y(n\Delta t)$ and $x^n=x(n\Delta t)$. With this notation condition (3) becomes $w^n=0$.

Modified Newton-Raphson method. The first step of our approach to the solution of Eq. (3) is a modified Newton-Raphson method, which includes the time evolution of the system.

Let us first consider w^1. We have

$$\begin{aligned} 0 &= w^1[(x^0,\mu),(x^0,\mu)] \\ &= w^1[(y^0,\nu),(x^0,\mu)] + (\delta y^0 \cdot \nabla_{y^0})w^1[(y^0,\nu),(x^0,\mu)] \\ &\quad + (\delta\nu \cdot \nabla_\nu)w^1[(y^0,\nu),(x^0,\nu)] + \cdots \end{aligned} \tag{4}$$

where $\delta y^n = x^n - y^n = -w^n$ and $\delta\nu = \mu - \nu$ [11]. For small Δt, we can write

$$w^1[(y^0,\nu),(x^0,\mu)] = w^0 + \Delta t[f(y^0,\nu) - f(x^0,\mu)] + \cdots. \tag{5}$$

Substituting Eq. (5) in the second and third terms of right-hand side of Eq. (4) and neglecting higher order terms, we get,

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$$w^1[(y^0,\nu),(x^0,\mu)] = w^0 - \Delta t[(\delta y^0 \cdot \nabla_{y^0}) f(y^0,\nu) - (\delta\nu \cdot \nabla_\nu) f(y^0,\nu)]. \quad (6)$$

Let W^n be a column matrix $(\delta w^n, -\delta\nu)^T$. It is convenient to write Eq. (6) in a matrix form as

$$W^1 = (I_{d+m} + \Delta t J^0) W^0 = A^0 W^0, \quad (7)$$

where I_k is the $k \times k$ identity matrix, $A^j = I_{d+m} + \Delta t J^j$, and the Jacobian matrix J^n is given by

$$J^n = \begin{pmatrix} (\nabla_{y^n} f(y^n,\nu))^T & (\nabla_\mu f(y^n,\nu))^T \\ 0 & I_m \end{pmatrix}. \quad (8)$$

Proceeding along similar lines, the equation for W^k is

$$W^k = (I + \Delta t J^{k-1}) W^{k-1} = A^{k-1} \cdots A^0 W^0. \quad (9)$$

Since by construction $W^0 = (0, \delta\nu)^T$, Eq. (9) gives us d independent linear equations for m unknown quantities $\delta\nu$ in terms of δy^k.

$$(W^k)_i = \sum_j (A^{k-1} \cdots A^0)_{ij} (W^0)_j, \quad i = 1, \ldots, d. \quad (10)$$

Thus, we get a full set of m equations by writing equations for $W^1, \cdots W^k$ so that $kd \geq m$. These equations can be solved by using some guess values for ν to yield $\delta\nu$. The process can be iterated as in Newton-Raphson method by taking the improved guess values as $\nu + \delta\nu$. The total duration of the time series required for this procedure is $k\Delta t$.

However, in practice, the above procedure works only for a small number of parameters. Due to the nonlinear nature of our equations, there are multiple solutions and as the number of unknown parameters increases, it is difficult to converge to the correct solution. A slower rate of convergence can partly avoid these problems if we iterate with the guess parameter values as $\nu + r\delta\nu$ where the moderating parameter r satisfies $0 < r \leq 1$. However, this procedure also fails as the number of unknown parameters increases further. The second step of our procedure addresses this problem.

Embedding with suitable time delays. We have noted above that Eq. (10) provides d independent equations for the unknown quantities $\delta\nu$. The total time duration is $k\Delta$ starting from the initial time $t=0$. Now, instead of iterating with time steps of Δt to obtain m equations, let us now fix some value of k and choose different initial times, say $t_1, \ldots, t_n$. For each initial time we obtain d equations with time evolution for $k\Delta t$ using Eq. (10). We get the required m equations provided $nd \geq m$. A simple way of choosing the initial times is by embedding with a delay of time τ so that the initial times may be $0, \tau, \ldots, (n-1)\tau$. The total time duration is $(n-1)\tau + k\Delta t$. Note that we need the values of the variables only at $0, \tau, \ldots, (n-1)\tau$ and $k\Delta t, \ldots, (n-1)\tau + k\Delta t$. There are different methods of choosing the time delay τ [12–14]. In our case, a much smaller time delay than the standard embedding time is sufficient to avoid the multiple solutions (see the examples).

How does the embedding avoid the problem of multiple solutions? Except the correct solution, the other (spurious) solutions depend on the local values of the variables and hence vary depending on the location of the variables in the phase space. By embedding we combine different phase-space locations and thus the spurious solutions are removed and only those invariant in the phase space survive. Numerical experiments show that after a suitable embedding almost always two solutions survive, the correct solution and the diverging solution. The diverging solution is easily identified and mostly can be avoided by changing the initial conditions.

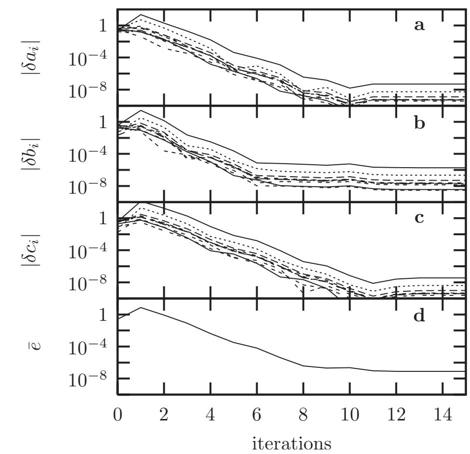

FIG. 1. (**a**) The absolute values of the errors in the estimation of the ten parameters, $|\delta a_i|, i=0,\ldots,9$ are shown as a function of the iterations of the method for Rössler system. These ten parameters are as in Eq. (11) for the dynamics of x_1 and correspond to all the terms with quadratic nonlinearity. **b** and **c** are similar plots for the errors $|\delta b_i|$ and $|\delta c_i|$ and are for the dynamics of x_2 and x_3, respectively. Rössler parameters are $a=b=0.2$ and $c=7.0$. The other parameters are $\Delta t=0.01$, $\tau=0.1$, $k=1$, and $r=1.0$. The initial guess values of the parameters were chosen to randomly lie between ± 1 of the correct values. **d**. The plot shows the average of the absolute errors $\bar{e}$ in the estimation of all the 30 parameters shown in **a–c** as a function of the iterations.

How does one avoid the exponential sensitivity of chaotic systems? The beauty of the Newton-Raphson method is that it works for both positive as well as negative slopes of the function at the roots of the equation. In the language of dynamical systems where the proposed modification is applied, it means that it works for both stable and unstable solutions.

We now illustrate the method described above using two examples. First consider the Rössler system $(\dot{x}_1, \dot{x}_2, \dot{x}_3) = [-x_2 - x_3, x_1 + a x_2, b + x_3(x_1 - c)]$. Let us rewrite the equations with all possible terms with quadratic nonlinearity. For x_1, we write

$$\dot{x}_1 = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_1^2 + a_5 x_2^2 + a_6 x_3^2 + a_7 x_1 x_2 + a_8 x_2 x_3 + a_9 x_3 x_1. \quad (11)$$

There are ten terms and ten parameters $a_0, \ldots, a_9$. We can write similar equations for x_2 and x_3 giving us twenty more parameters, say $b_0, \ldots b_9$ and $c_0, \ldots, c_9$. Comparing with the Rössler equations we see that the only nonzero parameters are $a_2 = a_3 = -1$, $b_1 = 1$, $b_2 = a$, $c_0 = b$, $c_3 = -c$, and $c_9 = 1$. Objective is to determine all the 30 parameters given the times series of the variables. Using the method described above it is possible to obtain all the parameters to a good accuracy. This is demonstrated in Fig. 1. Figure 1(a) shows how the

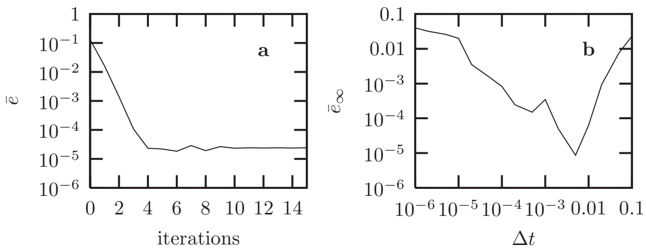

FIG. 2. (**a**) The plot shows the average of the absolute errors $\bar{e}$ in the estimation of all the ninety coupling constants as a function of the iterations for a network of ten coupled Rössler oscillators. The other parameters are as in Fig. 1. (**b**) The plot shows the asymptotic value of the average of the absolute errors $\bar{e}_\infty$ in the estimation of all the ninety parameters for a network of ten coupled Rössler oscillators.

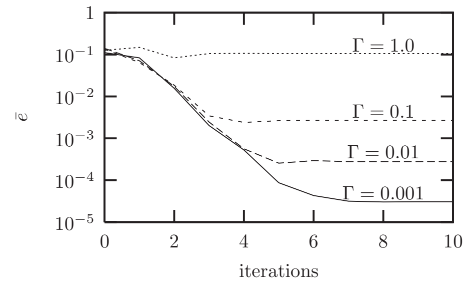

FIG. 3. The plot shows the average of the absolute errors $\bar{e}$ in the estimation of the 30 parameters of Rössler system expressed with all quadratic terms as a function of the iterations in the presence of noise. White noise of strength Γ is added to all the variables both at the times $t_i, i=1,\ldots,n$ and $t_i+k\Delta t$ which are used in Eq. (10). The parameters are as in Fig. 1 ($k=5$) and $n=10\,000$.

absolute values of the errors $|\delta a_i|, i=0,\ldots,9$ reduce to zero for the ten parameters a_i as a function of the iterations of our procedure. Figures 1(b) and 1(c) are similar plots for the twenty parameters b_i and c_i. Figure 1(d) shows the average of the absolute errors $\bar{e}=(1/30)\Sigma_i|\delta\nu_i|$ of all the 30 errors in Figs. 1(a)–1(c) as a function of the iterations. Thus, a reasonably accurate estimate of all the 30 parameters is obtained. Thus, for a class of three dimensional systems with quadratic nonlinearity, we are able to determine the exact form of the dynamical equations [15].

A comment about the accuracy of parameter estimation. We use Euler expansion in Eq. (5), and hence one may expect the error in parameter estimation to be of the order $(\Delta t)^2$. However, the actual parameter estimation is several orders better than this. This is because first, we use a more accurate procedure (fourth order Runge-Kutta) to obtain the variables y^n from y^{n-1} in Eq. (8) and then combine it with Euler expansion. Second, the evolution is nearly linear for the small time interval $k\Delta t$ between the initial and final values of the variables.

Another comment on the numerical procedure: in Fig. 1 the initial guess values of the parameters were chosen randomly to be within ± 1 of the correct values. It is possible to increase this range, but then one has to opt for a slower convergence, i.e., a smaller value of the moderating parameter r. E.g., if the initial guess values are chosen randomly to be within ± 5 of the correct values, then with $r=1$ we get numerical instabilities, but $r=0.01$ gives good results.

As the second example we consider a system of N coupled Rössler oscillators,

$$\dot{x}^{(i)} = f(x^{(i)}) + \sum_{j\neq i} A_{ij}(x_1^{(j)} - x_1^{(i)}), \quad i=1,\ldots,N \qquad (12)$$

Here, only the component x_1 is coupled. Here, the unknown parameters are the coupling constants $A_{ij}, i\neq j$. Again a reasonable estimate of the coupling constants is obtained using our method. This is shown in Fig. 2(a) which plots the average value of the absolute errors $\bar{e}$, in the estimation of all the 90 coupling constants for $N=10$, as a function of the iterations. Figure 2(b) shows the asymptotic value (for large iterates) of $\bar{e}_\infty$, as a function of the time interval Δt. The average error is decreases upto $\Delta t<0.005$ and then increases for $\Delta t>0.005$, till the system becomes unstable for $\Delta t>0.2$. This behavior is obtained as a competition between the Euler formula [Eq. (5)], which is better for smaller Δt and the information content of the data which is better for larger Δt.

We now turn to the important question of the effect of noise on the estimation of the parameters [16]. We note that the procedure described in steps A and B above is very sensitive to noise and fails to converge even with a small noise. Fortunately, it is possible to introduce a simple averaging procedure to take care of this problem.

Noise reduction. In step B above we choose different initial times, say $t_1,\ldots,t_n$, so that $nd\geq m$. We can increase the number of initial conditions, i.e., n, so that $nd\gg m$. Thus, we can generate a much larger number of equations for the unknown quantities $\delta\nu$ than needed for the calculations. This allows us to use the standard least square method of error minimization [9] and in the process a noise reduction.

We demonstrate the noise reduction technique by using the example of the 30 parameter estimation of Rössler system shown in Fig. 1. The noise is introduced in two different ways. First, it is introduced as an additive white noise in all the observed variables. Figure 3 shows the plot of the average value of the absolute errors $\bar{e}$ of the 30 parameters as a function of the iterates of our procedure in the presence of noise. For the noise strength $\Gamma\leq 0.2$, all the parameters get determined to a good accuracy i.e., $|\delta\nu_i|\ll\Gamma$. For $0.2<\Gamma\leq 1.0$, most of the parameters get $|\delta\nu_i|\ll\Gamma$, though for one or two parameters the error is of the order of Γ. For $\Gamma>1$, it is difficult to estimate all the 30 parameters, though some parameters can be estimated to a reasonable accuracy. Second, the noise is introduced as an additive term in all the evolution equations. Here, the parameter estimation is at least an order of magnitude better that the previous case (not shown in the figure). This is because the effect of noise in the variables is of the order $\Gamma\sqrt{\Delta t}$ which is much smaller than Γ. We have also carried out numerical simulations for other chaotic systems such as Lorenz system and find similar results as for Rössler system.

To conclude a method to estimate parameters of a nonlinear dynamical system from the time series data of the variables is introduced. As compared to the methods based on synchronization, the present method has two advantages.

First, the method works for any system and the concerned variables may or may not have the synchronizing property. Second, the total length of the time series required for the calculations is smaller than the synchronization based methods. Also the accuracy of parameter estimation is in general better. The method gives reasonably good results in the presence of noise. The method is demonstrated using two examples. Especially, it is possible to determine all the 30 parameters of a three dimensional nonlinear dynamical system with maximum quadratic nonlinearity. Thus, it is possible to determine the exact form of the dynamical equations for the class of systems with quadratic nonlinearity.

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